

Proof of Theorem 2

Observe that from the element-wise upper bound on \mathbf{H} , the following element-wise inequality holds $h_*\mathbf{E} \leq \mathbf{H} \leq h_*\mathbf{E} + \nu\mathbf{E}'$. Thus, from the linearity of $F(\mathbf{H}, \mathbf{S}) = \langle \mathbf{A}(\mathbf{S}), \mathbf{H} \rangle$ with respect to \mathbf{H} , we have that:

$$F(h_*\mathbf{E}, \mathbf{S}) \leq F(\mathbf{H}, \mathbf{S}) \leq F(h_*\mathbf{E} + \nu\mathbf{E}', \mathbf{S}),$$

where (by linearity) $F(h_*\mathbf{E} + \nu\mathbf{E}', \mathbf{S}) = h_*F(\mathbf{E}, \mathbf{S}) + \nu F(\mathbf{E}', \mathbf{S})$.

Next, employing terms: $a(\mathbf{S}) = F(\mathbf{E}, \mathbf{S}) = \langle H(\mathbf{S}), \mathbf{E} \rangle$ and $b(\mathbf{S}) = F(\mathbf{E}', \mathbf{S}) = \langle A(\mathbf{S}), \mathbf{E}' \rangle$, we may rewrite the bounds as:

$$h_*a(\mathbf{S}) \leq F(\mathbf{H}, \mathbf{S}) \leq h_*a(\mathbf{S}) + \nu b(\mathbf{S}).$$

Monotonicity:

The function $F(\mathbf{H}, \mathbf{S})$ is monotone with respect to \mathbf{S} if: $F(\mathbf{H}, \mathbf{S} \cup \{u\}) - F(\mathbf{H}, \mathbf{S}) \geq 0$. Applying the lower and upper bounds, we have that:

$$\begin{aligned} F(\mathbf{H}, \mathbf{S} \cup \{u\}) - F(\mathbf{H}, \mathbf{S}) &\geq h_*a(\mathbf{S} \cup \{u\}) - h_*a(\mathbf{S}) - \nu b(\mathbf{S}) \geq 0 \\ \implies \nu &\leq \frac{h_*a(\mathbf{S} \cup \{u\}) - h_*a(\mathbf{S})}{b(\mathbf{S})} = h_*\alpha(n, m) \end{aligned}$$

Thus, when the off-diagonal terms satisfy $h_{i,j} \leq h_*\alpha(n, m) \forall 0 \leq m \leq m_*, \forall (i, j) \in \mathbf{E}'$, we have that $F(\mathbf{H}, \mathbf{S})$ is monotone.

Submodularity:

The function $F(\mathbf{H}, \mathbf{S})$ is submodular with respect to \mathbf{S} if: $F(\mathbf{H}, \mathbf{S} \cup \{u\}) + F(\mathbf{H}, \mathbf{S} \cup \{v\}) \geq F(\mathbf{H}, \mathbf{S} \cup \{u, v\}) + F(\mathbf{H}, \mathbf{S})$. Again, applying the lower and upper bounds, we have that:

$$\begin{aligned} &F(\mathbf{H}, \mathbf{S} \cup \{u\}) + F(\mathbf{H}, \mathbf{S} \cup \{v\}) - F(\mathbf{H}, \mathbf{S} \cup \{u, v\}) - F(\mathbf{H}, \mathbf{S}) \\ &\geq h_*a(\mathbf{S} \cup \{u\}) + h_*a(\mathbf{S} \cup \{v\}) - h_*a(\mathbf{S} \cup \{u, v\}) - \nu b(\mathbf{S} \cup \{u, v\}) - h_*a(\mathbf{S}) - \nu b(\mathbf{S}) \geq 0 \\ \implies \nu &\leq h_* \frac{a(\mathbf{S} \cup \{u\}) + a(\mathbf{S} \cup \{v\}) - a(\mathbf{S} \cup \{u, v\}) - a(\mathbf{S})}{b(\mathbf{S} \cup \{u, v\}) + b(\mathbf{S})} = h_*\beta(n, m) \end{aligned}$$

Thus, when the off-diagonal terms satisfy $h_{i,j} \leq h_*\beta(n, m) \forall 0 \leq m \leq m_*, \forall (i, j) \in \mathbf{E}'$, we have that $F(\mathbf{H}, \mathbf{S})$ is submodular.

Proof of Corollary 3

Based on the diagonal dominance assumption on \mathbf{K} , it is clear that $\mathbf{E}' = \{i, j \in [n] \mid i \neq j\}$ indexes the off diagonal terms, and $\mathbf{E} = 1 - \mathbf{E}' = \mathbf{I}$. Given $\mathbf{A}(\mathbf{S})$ with entries $a_{i,j}(\mathbf{S}) = \frac{2}{n|\mathbf{S}|} \mathbf{1}_{[j \in \mathbf{S}]} - \frac{1}{|\mathbf{S}|^2} \mathbf{1}_{[i \in \mathbf{S}]} \mathbf{1}_{[j \in \mathbf{S}]}$, we can compute the bounds (8) simply by enumerating sums as:

$$\begin{aligned} a(\mathbf{S}) &= \langle \mathbf{A}(\mathbf{S}), \mathbf{I} \rangle = \frac{2m}{nm} - \frac{m}{m^2} = \frac{2}{n} - \frac{1}{m} \\ b(\mathbf{S}) &= \langle \mathbf{A}(\mathbf{S}), 1 - \mathbf{I} \rangle = \frac{2(nm - m)}{nm} - \frac{m^2 - m}{m^2} = \frac{2(n-1)}{n} - \frac{m-1}{m} \end{aligned}$$

Monotonicity: $J_p(\cdot)$ is monotone when the upper bound of the off-diagonal terms is given by $\alpha(n, m) = \frac{a(\mathbf{S} \cup \{u\}) - a(\mathbf{S})}{b(\mathbf{S})}$ by Theorem 2. We have that:

$$a(\mathbf{S} \cup \{u\}) - a(\mathbf{S}) = \frac{-1}{m+1} + \frac{1}{m}, \quad b(\mathbf{S}) = \frac{2(n-1)}{n} - \frac{m-1}{m}.$$

Thus:

$$\alpha(n, m) = \frac{n}{(m+1)(m(n-2) + n)}.$$

This is a decreasing function wrt m . Further, for the ground set $2^{[n]}$, we have that $m_* = n$, and $\alpha(n, n) = \frac{1}{n^2-1}$.

Submodularity: $J_p(\cdot)$ is submodular when the upper bound of the off-diagonal terms is given by $\beta(n, m) = \frac{a(\mathbf{S} \cup \{u\}) + a(\mathbf{S} \cup \{v\}) - a(\mathbf{S} \cup \{u, v\}) - a(\mathbf{S})}{b(\mathbf{S} \cup \{u, v\}) + b(\mathbf{S})}$ by Theorem 2. We have that:

$$\begin{aligned} a(\mathbf{S} \cup \{u\}) + a(\mathbf{S} \cup \{v\}) - a(\mathbf{S} \cup \{u, v\}) - a(\mathbf{S}) &= \frac{-2}{m+1} + \frac{1}{m+1} + \frac{1}{m} \\ b(\mathbf{S} \cup \{u, v\}) + b(\mathbf{S}) &= \frac{4(n-1)}{n} - \frac{m+1}{m+2} - \frac{m-1}{m} \end{aligned}$$

Thus:

$$\beta(n, m) = \frac{n}{(m+1)(n(m^2+3m+1)-2(m^2+2m))}$$

This is a decreasing function wrt m . Further, for the ground set $2^{[n]}$, we have that $m_* = n$, and $\beta(n, n) = \frac{1}{n^3+2n^2-2n-3}$.

Combined Bound: Finally, we show that $\beta(n, n) \leq \alpha(n, n)$, so that the bound $k_{i,j} \leq k_*\beta(n, n)$ is sufficient to guarantee both monotonicity and submodularity.

$$\begin{aligned} \beta(n, n) &\leq \alpha(n, n) \\ \implies \frac{1}{n^3+2n^2-2n-3} &\leq \frac{1}{n^2-1} \\ \implies n^2-1 &\leq n^3+2n^2-2n-3 \\ \implies 0 &\leq n^3+n^2-n-3 \\ \implies 0 &\leq (n-1)(n^2-2) \end{aligned}$$

which holds when $n > -1$ and $n \geq \sqrt{2}$. Thus $\beta(n, n) \leq \alpha(n, n)$. The proof is complete.

Proof of Theorem 7

A discrete function is linear if it can be written in the form $F(C) = \sum_{i \in [n]} w_i 1_{[i \in C]}$. Consider (9) and observe that:

$$\begin{aligned} L(C) &= \sum_{l \in C} \left| \frac{1}{n} \sum_{i \in [n]} k(x_i, x_l) - \frac{1}{m} \sum_{j \in S} k(x_j, x_l) \right| \\ &= \sum_{l \in [n]} \left(\left| \frac{1}{n} \sum_{i \in [n]} k(x_i, x_l) - \frac{1}{m} \sum_{j \in S} k(x_j, x_l) \right| \right) 1_{[l \in C]} \\ &= \sum_{l \in [n]} w_l 1_{[l \in C]}, \end{aligned}$$

where:

$$w_l = \left| \frac{1}{n} \sum_{i \in [n]} k(x_i, x_l) - \frac{1}{m} \sum_{j \in S} k(x_j, x_l) \right|.$$